Linear Algebra I
Problem Sheet 1: Vector Spaces

Exam style problems are marked by (*).

Vector Spaces

Question 1. Find the span of the following vectors: Decide whether these vectors are linearly dependent.

Question 2. (*) Vector spaces are constructed over fields. Describe an analogous construction of vector spaces over commutative rings. (Recall that rings are objects that generalise the integers, while fields are objects that generalise the rationals.)

Question 3. (*) Note that equations of the type

$$y = mx + b$$

only construct linear functions if $b = 0$. Why then are these called *linear equations*?

Bases

Question 4. Prove that the dual basis of a basis B of a vector space V is linearly independent.

Question 5. If $\{v_1, \ldots, v_n\}$ is a basis for a vector space V prove that every linearly independent set of vectors in V must have n or fewer elements.

Question 6. Let V be a finite dimensional vector space and let U and W be subspaces of V. Prove that

$$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W).$$

Question 7. (*) Consider the vector space V of polynomials of degree less than or equal to 2 defined over the real numbers.

- Construct the dual space V^*.
- Find a basis for V^*.
- Write the following element of V^* in the basis you found in part (ii):

$$f(a_0 + a_1 x + a_2 x^2) = a_0 + a_1 + a_2.$$